Two Queues with Alternating Service and Switching Times

O.J. Boxma, W.P. Groenendijk
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

This paper is concerned with a system of two queues, attended by a single server who alternately serves one customer of each queue (if not empty). The server experiences switching times in his transition from one queue to the other. It is shown that the joint stationary queue-length distribution, at the instants at which the server becomes available to a queue, can be determined via transformation to a Riemann boundary value problem. The latter problem can be completely solved for general service- and switching-time distributions. The stationary distributions of the waiting times at both queues, and of the cycle times of the server, are also derived. The results obtained, and in particular the extensive numerical data for moments of waiting times and cycle times, yield insight into the behavior of more general cyclic-service models. Such models are frequently used to analyse polling systems.

1. Introduction

Some 15 years ago the analysis of polling systems, employed to multiplex the service requests of several users in computer-terminal communication systems, gave rise to a new class of queueing models: a single server serves a number of queues in some cyclic fashion. Presently, these *cyclic-service* queueing models are finding a new application in local area networks with a ring or bus topology, employing a medium access control protocol based on token passing. Various service disciplines at the queues of the cyclic-service models have been considered, ranging from exhaustive service (when the server visits a queue, he serves its customers until the queue has become empty) to 1-limited service (when the server visits a queue, he serves only one customer, if present). For an extensive discussion of the many results that have recently been obtained for cyclic-service systems, we refer to the book [16] and survey paper [17] of TAKAGI. Generally speaking, the analysis of cyclic-service systems with exhaustive service is complex but tractable; 1-limited service, on the other hand, gives rise to very intricate mathematical problems, which only have been solved for models with not more than two queues.

The model with two queues, 1-limited service and no switching times has first been tackled in an important study of EISENBERG [8]. In the sequel this model with two queues, one server and 1-limited service discipline will be referred to as the *alternating-service model*. Eisenberg transformed the problem of determining the joint queue-length distribution at the two queues into the problem of solving a singular Fredholm integral equation. Almost simultaneously, FAYOLLE and IASNOGORODSKI in a highly original paper [9] solved another queueing problem with a two-dimensional state space, via transformation of the functional equation for the (bivariate) generating function of the joint queue-length distribution into a Riemann-Hilbert boundary value problem. Of course, the latter type of problem also belongs to the realm of singular integral equations. These two studies strongly stimulated the interest of J.W. Cohen in the analysis of queueing and random walk problems with a two-dimensional state space. He started an extensive research program which has led to a powerful method of transforming functional equations, encountered in such 'two-dimensional problems', into boundary value problems of the Riemann or Riemann-Hilbert type. One of the first fruits of this method has been a detailed analysis [6] of the alternating-service model without switching times. The resulting boundary value problem appeared to be a Dirichlet problem, a special case of a Riemann-Hilbert problem. In [1] the analysis was extended to the case *with* switching times of the server between queues, but the restriction was made that both queues had identical characteristics. This time, the resulting boundary value problem was a Riemann-Hilbert problem.

The goal of the present paper is to give an exact analysis of the alternating-service model with switching times, with arbitrary service-time distributions and switching-time distributions. As in the just mentioned studies, the arrival processes at both queues are independent Poisson processes.

The organization of the paper is as follows. In Section 2 the model is described in detail. Section 3 contains the main analysis. The joint stationary queue-length distribution, at the instants at which the server becomes available to a queue, is determined via transformation into a Riemann boundary value problem. Once the joint queue-length distribution is known, one can easily derive expressions for various important performance measures, like waiting times of customers and cycle times of the server (the time between two successive arrivals of the server at a particular queue). Waiting times are studied in Section 4, with particular attention to the mean waiting times; cycle times are studied in Section 5, with particular attention to second moments of the cycle times (first moments of cycle times are trivially determined). Section 6 is devoted to a numerical evaluation of some important performance measures of the alternating-service model. It is shown that the boundary value problem formulation leads to formulas which can be numerically evaluated in a straightforward manner. The numerical results in this section may also provide additional insight into the behavior of the alternating-service model and, more generally, of cyclic-service models. E.g., it is shown that, while first moments of cycle times do not depend on the number of the queue at which the cycle starts, second moments of cycle times generally differ only slightly. This supports an approximation assumption in [2,3], to the effect that the second moments are equal.

2. Model description

A single server S serves two queues Q_1, Q_2 (with infinite buffer capacities) in cyclic order. The arrival process of customers at Q_i is a Poisson process with rate λ_i, $i=1,2$. The service times at Q_i are independent, identically distributed stochastic variables with distribution $B_i(.)$, with first moment β_i, second moment $\beta_i^{(2)}$ and LST (Laplace-Stieltjes Transform) $\beta_i(\cdot)$. The various arrival and service processes are independent.

The utilization at Q_i, ρ_i, is defined as

$$\rho_i := \lambda_i \beta_i, \quad i=1,2. \tag{2.1}$$

The total utilization of the server, ρ, is defined as

$$\rho := \rho_1 + \rho_2. \tag{2.2}$$

The server serves one customer, if any, from Q_1, and after a switching time he inspects Q_2. He serves one customer, if any, from Q_2, and switches back to Q_1; etc. The successive switching times from Q_i to $Q_{(i+1) \bmod 2}$ are independent, identically distributed stochastic variables, also independent of the service times, with distribution $S_i(\cdot)$. Their first moment, second moment and LST are respectively denoted by s_i, $s_i^{(2)}$ and $\sigma_i(.)$.

Let $\mathbf{C}_i$ denote the time between two successive arrivals of S at Q_i, the *cycle time* for Q_i. Clearly each cycle consists of two switches and at most one service at each of the two queues. The first and second moments of the total switching time during one cycle are respectively denoted by

$$s := s_1 + s_2, \tag{2.3}$$

$$s^{(2)} := s_1^{(2)} + 2s_1s_2 + s_2^{(2)}. \tag{2.4}$$

It is well known, and easily seen (cf. WATSON [19]) that the mean cycle time $E\mathbf{C}_i$ is independent of i, and is given by

$$E\mathbf{C} = \frac{s}{1-\rho}. \tag{2.5}$$

Ergodicity conditions

In the model *without* switching times, $\rho<1$ is easily seen to be a necessary and sufficient condition for ergodicity. KUEHN [14] has shown that, in the model *with* switching times,

$$\rho + \max(\lambda_1 s, \lambda_2 s) < 1, \tag{2.6}$$

is a necessary condition for ergodicity (indeed, the mean number of arrivals at Q_i during a cycle, $\lambda_i s/(1-\rho)$, should be less than one). SZPANKOWSKI and REGO [15] have recently proved that this condition is also sufficient. In the sequel we assume that the system is ergodic, hence necessarily (2.6) holds.

3. FORMULATION AND SOLUTION OF THE BOUNDARY VALUE PROBLEM

Let $\mathbf{q}_i^{(1)}$, $i=1,2$, denote the number of customers at Q_i at those instants at which S arrives at Q_1; similarly for $\mathbf{q}_i^{(2)}$. Let

$$F_j(z_1,z_2) := E[z_1^{\mathbf{q}_1^{(j)}} z_2^{\mathbf{q}_2^{(j)}}], \quad |z_1|\leq 1, \; |z_2|\leq 1, \; j=1,2. \tag{3.1}$$

Our goal in this section is to determine $F_1(z_1,z_2)$ and $F_2(z_1,z_2)$. This goal will be accomplished by formulating and solving a so-called Riemann boundary value problem (cf. GAKHOV [12]). Once $F_i(z_1,z_2)$ is determined, the LST of the waiting-time distribution at Q_i and of the distribution of the cycle time $\mathbf{C}_i$ can be obtained. In Sections 4 and 5 we will demonstrate this, deriving mean waiting times and second moments of cycle times as by-products.

The vector of queue lengths at Q_1 and Q_2 at successive arrival epochs of server S at a queue forms a vector Markov chain. A study of its transition probabilities yields the following recurrence relations for the generating functions $F_1(z_1,z_2)$ and $F_2(z_1,z_2)$: for $|z_1|\leq 1$, $|z_2|\leq 1$,

$$F_2(z_1,z_2) = \{[F_1(z_1,z_2) - F_1(0,z_2)]\, z_1^{-1}\beta_1(x) + F_1(0,z_2)\}\, \sigma_1(x), \tag{3.2}$$

$$F_1(z_1,z_2) = \{[F_2(z_1,z_2) - F_2(z_1,0)]\, z_2^{-1}\beta_2(x) + F_2(z_1,0)\}\, \sigma_2(x), \tag{3.3}$$

with for $|z_1|\leq 1$, $|z_2|\leq 1$:

$$x := \lambda_1(1-z_1) + \lambda_2(1-z_2). \tag{3.4}$$

Substitution of (3.2) into (3.3) yields:

$$\begin{aligned} K(z_1,z_2)F_1(z_1,z_2) = {} & F_1(0,z_2)\{\beta_2(x)\,\sigma_1(x)\,\sigma_2(x)\,(z_1 - \beta_1(x))\} + \\ & F_2(z_1,0)\{z_1\,\sigma_2(x)\,(z_2 - \beta_2(x))\}, \quad |z_1|\leq 1, \; |z_2|\leq 1, \end{aligned} \tag{3.5}$$

while analogously,

$$\begin{aligned} K(z_1,z_2)F_2(z_1,z_2) = {} & F_2(z_1,0)\{\beta_1(x)\,\sigma_1(x)\,\sigma_2(x)\,(z_2 - \beta_2(x))\} + \\ & F_1(0,z_2)\{z_2\,\sigma_1(x)\,(z_1 - \beta_1(x))\}, \quad |z_1|\leq 1, \; |z_2|\leq 1; \end{aligned} \tag{3.6}$$

here $K(z_1,z_2)$ is the *kernel of the functional equation,* defined as

$$K(z_1,z_2) := z_1 z_2 - \beta_1(x)\,\beta_2(x)\,\sigma_1(x)\,\sigma_2(x). \tag{3.7}$$

Relation (3.5) is the starting point for our analysis. The main idea is similar to the one in [6,7] for the model without switching times and in [1] for the model with switching times and with identical characteristics of both queues ('the symmetric model'): the determination of $F_1(z_1,z_2)$ from (3.5) will be reduced to the solution of a boundary value problem (BVP) of mathematical physics. In the model without switching times, this BVP was a Dirichlet problem, a special case of a Riemann-Hilbert problem; in the symmetric model with switching times, this BVP was a Riemann-Hilbert problem; and in the present more general model, reduction to a Riemann problem can be accomplished. In fact, in the latter two cases both a Riemann-Hilbert and a Riemann problem formulation are possible; the Riemann formulation seems to be somewhat more natural here. The analysis consists of four steps.

Step 1: the set-up
According to its definition as a generating function, $F_1(z_1,z_2)$ should be regular for $|z_1| < 1$, continuous for $|z_1| \leqslant 1$, for every fixed z_2 with $|z_2| \leqslant 1$; and similarly with z_1 and z_2 interchanged. Hence every zero (z_1,z_2) of the kernel $K(z_1,z_2)$ in (3.5) should be a zero of the right-hand side of (3.5). This condition must lead to the yet unknown functions $F_1(0,z_2)$ and $F_2(z_1,0)$ in the right-hand side of (3.5), and hence to $F_1(z_1,z_2)$.

Step 2: analysis of the kernel
It is not possible to determine, explicitly, exactly one zero z_1 in $|z_1| \leqslant 1$ for each z_2 in $|z_2| \leqslant 1$. Various sets of zero-pairs of the kernel $K(z_1,z_2)$ can be determined; our choice will lead to a Riemann BVP. $K(z_1,z_2)$ is a so-called Poisson kernel (cf. Ch. II.4 of [7]). It has the same structure as the Poisson kernel defined in (2.4) on p. 274 of [7], where the alternating-service model *without* switching times is studied. Therefore we can proceed as in [7] (cf. also [1]). First introduce

$$\beta(x) := \beta_1(x)\, \beta_2(x)\, \sigma_1(x)\, \sigma_2(x),$$

$$\lambda := \lambda_1 + \lambda_2,$$

$$r_1 := \lambda_1/\lambda, \qquad r_2 := \lambda_2/\lambda.$$

Without loss of generality, it will henceforth be assumed that

$$r_1 \geqslant r_2.$$

Finally introducing

$$w_1 := 2r_1 z_1, \qquad w_2 := 2r_2 z_2,$$

we can rewrite $4r_1r_2K(z_1,z_2)$ as

$$w_1 w_2 - 4r_1r_2\beta(\lambda(1-(w_1+w_2)/2)).$$

The symmetry of this expression suggests to look for pairs of zeros of the kernel that are each other's complex conjugates: $(w_1,w_2)=(w,\bar{w})$. These pairs of zeros turn out to supply all the information we need. The following should hold for w:

$$|w|^2 = 4r_1r_2\beta(\lambda(1- \operatorname{Re} w)).$$

Write

$$w = e^{i\phi}2\sqrt{r_1r_2}\,\sqrt{\beta(\lambda(1- \operatorname{Re} w))}, \qquad 0 \leqslant \phi \leqslant 2\pi.$$

Defining $\delta(\phi)$ to be the unique zero of

$$\delta - 2\sqrt{r_1r_2}\cos(\phi)\sqrt{\beta(\lambda(1-\delta))}, \qquad 0 \leqslant \phi \leqslant 2\pi,\ \operatorname{Re}\delta \leqslant 1, \tag{3.8}$$

it is seen that, when ϕ once traverses the trajectory $[0,2\pi]$,

$$w = w(\phi) = \delta(\phi)(1+i\,\tan(\phi))$$

once encircles a simple, smooth contour F that is contained in the unit circle.

F is an egg-shaped contour. Using the notation L^+ (L^-) for the interior (exterior) of a contour L, we have $0 \in F^+$. Every $w \in F$ satisfies the relation $|w|^2 = 4r_1 r_2 \beta(\lambda(1 - \text{Re}\, w))$; hence $(z_1, z_2) = (w/2r_1, \bar{w}/2r_2)$ forms a pair of zeros of the kernel $K(z_1, z_2)$ for every $w \in F$.

Step 3: formulation of a Riemann boundary value problem

The choice of zero-pairs $(z_1, z_2) = (w/2r_1, \bar{w}/2r_2)$ of the kernel leads, in a natural way, to the formulation of a Riemann BVP. In the formulation and solution of the BVP a few technical difficulties will arise; these are mainly related to the position of the point $2r_2$ with respect to the contour F. Depending on the choice of parameters, this point can be inside, on or outside the contour. For the sake of clarity, we restrict ourselves here to the case $2r_2 \in F^+$; see Remark 3.1 for a short discussion of the case $2r_2 \in F$ (which occurs, e.g., for $r_2 = 1/2$) and the (relatively rare) case $2r_2 \in F^-$.

Basically, the Riemann BVP amounts to finding two functions, one regular inside a certain smooth contour and the other one regular outside that contour, such that a certain linear relation exists between these functions on the contour; see [7] for a short exposition, and see GAKHOV [12] for a detailed discussion. The first part of Step 3 concerns that linear relation between two functions on a contour. The right-hand side of (3.5) should be zero for all those $w \in F$, for which $(w/2r_1, \bar{w}/2r_2)$ forms a pair of zeros of $K(z_1, z_2)$ inside the product of unit circles. Now $|w/2r_1| \leq 1$ always holds for $w \in F$, but the possibility that $|\bar{w}/2r_2| > 1$ cannot be excluded. Fortunately, in the latter case analytic continuation can be used (cf. below (3.21)) to show that, for *all* $w \in F$, the right-hand side of (3.5) should be zero. So for *all* $w \in F$, the following linear relation should exist between $F_1(0, \bar{w}/2r_2)$ and $F_2(w/2r_1, 0)$:

$$F_1(0,\bar{w}/2r_2)\,[\beta_2(\lambda(1-\text{Re}\,w))\,\sigma_1(\lambda(1-\text{Re}\,w))\,\sigma_2(\lambda(1-\text{Re}\,w)) \times \{\frac{w}{2r_1} - \beta_1(\lambda(1-\text{Re}\,w))\}] + F_2(w/2r_1,0)\,[\frac{w}{2r_1}\sigma_2(\lambda(1-\text{Re}\,w))\,\{\frac{\bar{w}}{2r_2} - \beta_2(\lambda(1-\text{Re}\,w))\}] = 0. \tag{3.9}$$

Hence

$$F_2(w/2r_1, 0) = G(w)\, F_1(0, \bar{w}/2r_2), \quad w \in F, \tag{3.10}$$

with

$$G(w) := -\beta_2(\lambda(1-\text{Re}\,w))\,\sigma_1(\lambda(1-\text{Re}\,w))\,\frac{1}{w/2r_1} \times \frac{w/2r_1 - \beta_1(\lambda(1-\text{Re}\,w))}{\bar{w}/2r_2 - \beta_2(\lambda(1-\text{Re}\,w))}. \tag{3.11}$$

In the standard formulation of the Riemann BVP, the involved smooth contour is the unit circle. A conformal mapping of F^+ onto the interior C^+ of the unit circle C will lead us to a standard Riemann BVP. The second part of

Step 3 concerns this conformal mapping:

$$z = f(w):\ F^+ \to C^+, \tag{3.12}$$

and its inverse, the conformal mapping

$$w = f_0(z):\ C^+ \to F^+. \tag{3.13}$$

One can write (cf. GAIER [11], Section 2.1; see also Section I.4.4 of [7]):

$$f_0(z) = z\ \exp[\frac{1}{2\pi}\int_0^{2\pi} \log\{\frac{\delta(\theta(\omega))}{\cos(\theta(\omega))}\}\ \frac{e^{i\omega}+z}{e^{i\omega}-z}\, d\omega], \qquad |z|<1, \tag{3.14}$$

with the angular deformation $\theta(\cdot)$ being uniquely determined as the continuous solution of the *Theodorsen* integral equation

$$\theta(\phi) = \phi - \frac{1}{2\pi}\int_0^{2\pi} \log\{\frac{\delta(\theta(\omega))}{\cos(\theta(\omega))}\}\ \mathrm{cotan}\{\frac{1}{2}(\omega-\phi)\}d\omega, \qquad 0\leqslant\phi\leqslant 2\pi; \tag{3.15}$$

$\theta(\phi)$ is a strictly increasing and continuous function of ϕ, and $\theta(\phi)=2\pi-\theta(2\pi-\phi)$. According to the corresponding-boundaries theorem ([7], p. 66), $f_0(z)$ is continuous in $C^+ \cup C$. Application of the conformal mapping $f_0(\cdot)$ transforms (3.10) into:

$$F_2(f_0(z)/2r_1,0) = G(f_0(z))\, F_1(0,f_0(1/z)/2r_2), \qquad z \in C. \tag{3.16}$$

Introducing the functions

$$\hat{F}_2(z) := F_2(f_0(z)/2r_1,0), \qquad z \in C^+ \cup C, \tag{3.17}$$

$$\hat{F}_1(z) := F_1(0,f_0(1/z)/2r_2), \qquad z \in C \cup C^-, \tag{3.18}$$

$$H(z) := G(f_0(z)) = -\beta_2(\lambda(1-\mathrm{Re}\, f_0(z)))\, \sigma_1(\lambda(1-\mathrm{Re}\, f_0(z)))\times \tag{3.19}$$

$$\frac{1}{f_0(z)/2r_1}\ \frac{f_0(z)/2r_1-\beta_1(\lambda(1-\mathrm{Re}\, f_0(z)))}{f_0(1/z)/2r_2-\beta_2(\lambda(1-\mathrm{Re}\, f_0(z)))}, \qquad z \in C,$$

(3.16) can be rewritten as:

$$\hat{F}_2(z) = H(z)\, \hat{F}_1(z), \qquad z \in C. \tag{3.20}$$

There are some technical requirements for a Riemann BVP formulation: $H(z)$ should satisfy a so-called Hölder condition on C (this can be easily verified, and will not be further discussed); and $0<|H(z)|<\infty$ for $z \in C$. The third part of Step 3 concerns a proof that $H(z)\neq 0$ for $z \in C$ (the proof that $|H(z)|<\infty$ is left to the reader). We prove the equivalent statement that $G(w)\neq 0$ for $w \in F$. The two points of F on the real axis are the only candidate zeros of $G(w)$, $w \in F$. It is soon clear that we can concentrate on $w=\delta(0) \in F$, and that it remains to show that $\delta(0)/2r_1 - \beta_1(\lambda(1-\delta(0))) \neq 0$. The assumption that $2r_2 \in F^+$ implies that $\delta(0)/2r_2 > \beta_2(\lambda(1-\delta(0)))$. The definition of $\delta(0)$, see (3.8), implies that

$$\delta(0) = 2\sqrt{r_1 r_2}\,\sqrt{\beta(\lambda(1-\delta(0)))} < 2\sqrt{r_1 r_2}\,\sqrt{\beta_1(\lambda(1-\delta(0)))}\,\sqrt{\beta_2(\lambda(1-\delta(0)))}.$$

Hence

$$\delta(0)/2r_1 \geqslant \beta_1(\lambda(1-\delta(0))) \quad \text{and} \quad \delta(0)/2r_2 \geqslant \beta_2(\lambda(1-\delta(0)))$$

are not simultaneously possible. In view of the above,

$$\delta(0)/2r_1 < \beta_1(\lambda(1-\delta(0))). \tag{3.21}$$

We are almost ready to formulate our Riemann BVP. It remains to show that $\hat{F}_1(z)$ is regular for $z \in C^-$, continuous for $z \in C \cup C^-$, and $\hat{F}_2(z)$ is regular for $z \in C^+$, continuous for $z \in C^+ \cup C$. We show, equivalently, that $F_2(w/2r_1,0)$ and $F_1(0,w/2r_2)$ are regular in F^+ and continuous in $F^+ \cup F$. Since by assumption $2r_1 \geqslant 1$, while F is contained in the unit circle, it immediately follows that $F_2(w/2r_1,0)$ is regular in F^+, and continuous in $F^+ \cup F$. It is somewhat more difficult to show that $F_1(0,w/2r_2)$ is also regular in F^+ and continuous in $F^+ \cup F$. First note that $\delta(0)=\max|w|$, $w \in F^+ \cup F$. Subsequently note that $F_1(0,\delta(0)/2r_2)$ is finite, because $F_2(\delta(0)/2r_1,0)/G(\delta(0))$ is finite. These observations, combined with the fact that the coefficients in the series expansion of $F_1(0,w/2r_2)$ are non-negative, lead to the stated regularity and continuity properties of $F_1(0,w/2r_2)$ (cf. [7], p. 277, for a similar reasoning).

We have now arrived at a standard, homogeneous, Riemann BVP on the unit circle:
Determine two functions $\hat{F}_1(z)$ and $\hat{F}_2(z)$, such that

- (3.20) holds, with $H(.)$ satisfying a Hölder condition on C and $0<|H(z)|<\infty$, $z \in C$;
- $\hat{F}_1(z)$ is regular for $z \in C^-$, continuous for $z \in C \cup C^-$;
- $\hat{F}_2(z)$ is regular for $z \in C^+$, continuous for $z \in C^+ \cup C$;
- $\hat{F}_1(z) \to A$ for $|z| \to \infty$, with A a constant.

Step 4: solution of the Riemann boundary value problem
A crucial role in the solution of the Riemann BVP is played by the *index*, χ, of the function $H(.)$ on C. This index is by definition:

$$\chi := \text{ind}_{z \in C}\, H(z) = \frac{1}{2\pi} \int_{z \in C} d\{\arg H(z)\} \tag{3.22}$$

$$= \text{ind}_{w \in F}\, G(w) = \frac{1}{2\pi} \int_{w \in F} d\{\arg G(w)\}.$$

LEMMA 3.1. $\chi = 0$ *for* $2r_2 \in F^+$.

PROOF. From (3.11),

$$\chi = -\text{ind}_{w \in F} \frac{w}{2r_1} + \text{ind}_{w \in F}\, [w/2r_1 - \beta_1(\lambda(1-\text{Re}\, w))] \tag{3.23}$$

$$-\text{ind}_{w \in F}\, [\bar{w}/2r_2 - \beta_2(\lambda(1-\text{Re}\, w))] =$$

$$-1 + \mathrm{ind}_{w \in F} [w/2r_1 - \beta_1(\lambda(1-\mathrm{Re}\, w))]$$

$$+\mathrm{ind}_{w \in F} [w/2r_2 - \beta_2(\lambda(1-\mathrm{Re}\, w))].$$

The fact that $2r_2 \in F^+$ implies that $w/2r_2 > \beta_2(\lambda(1-\mathrm{Re}\, w))$ for $w=\delta(0) \in F$. It now readily follows that

$$\mathrm{ind}_{w \in F} [w/2r_2 - \beta_2(\lambda(1-\mathrm{Re}\, w))] = 1. \tag{3.24}$$

Similarly, from (3.21)

$$\mathrm{ind}_{w \in F} [w/2r_1 - \beta_1(\lambda(1-\mathrm{Re}\, w))] = 0. \tag{3.25}$$

The lemma follows from (3.23), (3.24) and (3.25). □

The homogeneous Riemann BVP formulated at the end of Step 3, with index 0, has the following solution (cf. [7], Section I.2.3):

$$\hat{F}_1(z) = A \exp[\frac{1}{2\pi i} \int_{t \in C} \frac{\log H(t)}{t-z} dt], \quad z \in C^-, \tag{3.26}$$

$$\hat{F}_2(z) = A \exp[\frac{1}{2\pi i} \int_{t \in C} \frac{\log H(t)}{t-z} dt], \quad z \in C^+, \tag{3.27}$$

with $A = F_1(0,0)$ yet to be determined.
Formulas (3.26) and (3.27) lead, in combination with (3.17), (3.18) and the definition (3.12) of the conformal mapping $f(.)$, to our main result:

THEOREM 3.1.

$$F_1(0,w/2r_2) = A \exp[\frac{1}{2\pi i} \int_{t \in C} \frac{\log H(t)}{t - 1/f(w)} dt], \quad w \in F^+, \tag{3.28}$$

$$F_2(w/2r_1,0) = A \exp[\frac{1}{2\pi i} \int_{t \in C} \frac{\log H(t)}{t - f(w)} dt], \quad w \in F^+. \tag{3.29}$$

It remains to determine the constant $A = F_1(0,0)$. Substitution of $w=2r_2$ in (3.28) yields a linear relation between $F_1(0,1)$ and $F_1(0,0)$. $F_1(0,1)$ can be determined in various ways. E.g., substituting $z_2=1$ in (3.5) and subsequently letting $z_1 \to 1$ gives one linear relation between $F_1(0,1)$ and $F_2(1,0)$; applying a similar procedure to (3.6) gives a second linear relation between those quantities. Solution of the two equations yields:

$$F_1(0,1) = 1 - \frac{\lambda_1 s}{1-\rho}, \tag{3.30}$$

$$F_2(1,0) = 1 - \frac{\lambda_2 s}{1-\rho}. \tag{3.31}$$

The following observation also immediately leads to (3.30) (and similarly (3.31)): $1-F_1(0,1)$ is the probability that server S finds Q_1 not empty upon his arrival. Therefore it equals the fraction of times that S serves a customer in Q_1 during his visit. By a balance argument, this fraction also equals the mean number of arrivals at Q_1 during a cycle of the server. According to (2.5), the mean cycle time of the server equals $s/(1-\rho)$; hence the mean number of arrivals at Q_1 during a cycle of the server equals $\lambda_1 s/(1-\rho)$.

The above implies that the constant A is given by:

$$A = F_1(0,0) = (1-\frac{\lambda_1 s}{1-\rho}) \exp[-\frac{1}{2\pi i} \int_{t \in C} \frac{\log H(t)}{t-1/f(2r_2)} dt]. \tag{3.32}$$

REMARK 3.1. Earlier, the restrictive assumption $2r_2 \in F^+$ has been made. However, the cases $2r_2 \in F$ and $2r_2 \in F^-$ can also occur. E.g., if $r_1 = r_2 = 1/2$, then $2r_2 = 1 = \delta(0) \in F$; examples in which $2r_2 \in F^-$ can also be constructed, although one has to be careful not to violate the ergodicity condition (cf. [7], pp. 360-361). We shortly consider Steps 3 and 4 above for these two cases.

(i) $2r_2 \in F$.
The Riemann BVP formulation and the proof that the index $\chi=0$ proceed as before. The special case $r_1 = r_2 = 1/2$ requires some subtlety. Now $\delta(0)=1$; both the numerator and denominator of the right-hand side of (3.11) become zero, but the zeros cancel and again $G(\delta(0)) \neq 0$. Furthermore, (3.23) reduces to $\chi = -1+1/2+1/2=0$. For all cases in which $2r_2 \in F$, the solution of the Riemann BVP proceeds as before, and Theorem 3.1 still holds. A minor difficulty is that $F_1(0,1)$ cannot be obtained from (3.28) by substitution of $w=2r_2$. But application of the so-called *Plemelj-Sokhotski* formula (cf. [7], Formula (I.1.6.4)) to (3.28) leads to an expression for $F_1(0,w/2r_2)$, $w \in F$. In the resulting expression, the substitution $w=2r_2$ can be made.

(ii) $2r_2 \in F^-$.
The formulation of the Riemann BVP proceeds as before. This time verification of the regularity of $F_1(0,w/2r_2)$ in F^+ is trivial. Again the index $\chi=0$, but verification is less straightforward. It is based on the observation that, for $2r_2 \in F^-$ and $w=\delta(0) \in F$, the coefficients of the functions $w/2r_1 - \beta_1(\lambda(1-\text{Re } w))$ and $\bar{w}/2r_2 - \beta_2(\lambda(1-\text{Re } w))$ in (3.9) are both non-negative, so that these functions have opposite signs. Theorem 3.1 still holds, but a major difficulty now is that $F_1(0,1)$ cannot be obtained by substitution of $w=2r_2$ in (3.28). One might obtain an analytic continuation of $F_1(0,w/2r_2)$ in F^-, but this is numerically impractical. A numerically feasible approach is to obtain numerical values for, say, $F_1(0,1)$ by using a Taylor series expansion of $F_1(0,z)$ around $z=0$. Chapter IV.1 of [7] contains an example of this procedure, for the alternating-service model without switching times.

Expressions for the generating functions $F_1(z_1,z_2)$ and $F_2(z_1,z_2)$ follow from (3.5), (3.6) and Theorem 3.1. In the next two sections we use the results obtained about queue-length generating functions to derive information about the distributions of waiting times and cycle times, and in particular about their moments.

4. Waiting times

In this section we shall derive an expression for $E\mathbf{w}_2$, the mean waiting time at Q_2. $E\mathbf{w}_2$ will be expressed in some given model parameters and in the function $d/dz\, F_1(0,z)$, evaluated at $z=1$. The latter function is obtained from (3.28) after differentiation with respect to w and substitution of $w=2r_2$. $E\mathbf{w}_1$ cannot be obtained from (3.29) in a similar way; the fact that $2r_1 \in F^-$ poses a problem. To reach the point $2r_1$ out of F^+ we might take our refuge to analytic continuation, but this would lead to numerical difficulties. We might also use a Taylor series expansion of $F_2(w/2r_1,0)$ around $w=0$; such an approach would also be useful in case $2r_2 \in F^-$, as discussed at the end of the previous section.

Anyway, once we have calculated $E\mathbf{w}_2$, $E\mathbf{w}_1$ can be obtained from a linear relation between $E\mathbf{w}_1$ and $E\mathbf{w}_2$. One can derive such a relation from (3.1) and (3.2) by repeated differentiation, using (4.2) below. Watson [19] has done this, more generally, for a single-server, multi-queue system with cyclic service. He has thus derived a linear relation between the mean waiting times at the various queues. For two queues, his result reduces to:

$$\rho_1[1-\frac{\lambda_1 s}{1-\rho}]E\mathbf{w}_1 + \rho_2[1-\frac{\lambda_2 s}{1-\rho}]E\mathbf{w}_2 \tag{4.1}$$
$$= \rho\frac{\lambda_1\beta_1^{(2)}+\lambda_2\beta_2^{(2)}}{2(1-\rho)} + \rho\frac{s^{(2)}}{2s} + \frac{s}{2(1-\rho)}[\rho^2+\rho_1^2+\rho_2^2].$$

See [4,5] for a generalization, with a probabilistic proof, of Watson's result to the case of a single-server, multi-queue system with cyclic service and a mixture of various types of service disciplines at the queues.

$E\mathbf{w}_2$ is obtained in the following way. By a standard M/G/1-type argument (cf. Watson [19], Takagi [16]) we can write:

$$E\{e^{-\lambda_2(1-z)\mathbf{w}_2}\} = \frac{F_2(1,z) - F_2(1,0)}{z(1-F_2(1,0))}. \tag{4.2}$$

Indeed, the customers present in Q_2 at the start of a non-empty service period at that queue, excluding the customer about to be served, are just the customers who had arrived during the waiting time of that customer. Note that (4.2) completely determines the waiting-time distribution at Q_2; a similar relation holds for the transform of the waiting-time distribution at Q_1. From (4.2) and (3.31),

$$E\mathbf{w}_2 = \frac{1-\rho}{\lambda_2^2 s}\{\frac{d}{dz}F_2(1,z)\}_{z=1} - \frac{1}{\lambda_2}. \tag{4.3}$$

The derivative occurring in (4.3) follows from (3.6) after a tedious but straightforward calculation. Denote by β and $\beta^{(2)}$ the first and second moments of the sum of a service time at Q_1, a switching time from Q_1 to Q_2, a service time at Q_2 and a switching time from Q_2 to Q_1. Then

$$\{\frac{d}{dz}F_2(1,z)\}_{z=1} = (1-\frac{\lambda_2 s}{1-\rho})\left[\frac{\lambda_2(\beta_1+s)(1-\lambda_2\beta_2)}{1-\lambda_2\beta} - \frac{\frac{1}{2}\lambda_2^2\beta_2^{(2)}}{1-\lambda_2\beta} + (1-\lambda_2\beta_2)\frac{\frac{1}{2}\lambda_2^2\beta^{(2)}}{(1-\lambda_2\beta)^2}\right] - \{\frac{d}{dz}F_1(0,z)\}_{z=1}\frac{\lambda_2\beta_1}{1-\lambda_2\beta} - (1-\frac{\lambda_1 s}{1-\rho})\left[\frac{\lambda_2\beta_1(1+\lambda_2 s_1)+\frac{1}{2}\lambda_2^2\beta_1^{(2)}}{1-\lambda_2\beta} + \frac{\frac{1}{2}\lambda_2^3\beta_1\beta^{(2)}}{(1-\lambda_2\beta)^2}\right]. \tag{4.4}$$

It remains to determine the derivative occurring in the right-hand side of (4.4). From (3.28),

$$\{\frac{d}{dz}F_1(0,z)\}_{z=1} = 2r_2\{\frac{d}{dw}F_1(0,w/2r_2)\}_{w=2r_2} \tag{4.5}$$

$$= 2r_2\; F_1(0,1)\,\frac{-f^{(1)}(2r_2)}{(f(2r_2))^2}\,\frac{1}{2\pi i}\int_{t\in C}\frac{\log H(t)}{(t-1/f(2r_2))^2}\,dt.$$

$E\mathbf{w}_2$ follows from (4.3), (4.4), (4.5) and (3.30). As indicated above, $E\mathbf{w}_1$ subsequently follows from (4.1).

5. Cycle times

In Section 2 the cycle time $E\mathbf{C}_i$ for Q_i has been defined as the time between two consecutive arrivals of S at Q_i. Both from a theoretical and a practical point of view, cycle times are important quantities in cyclic-service systems. Mean cycle times are easily calculated (cf. (2.5)), but in cyclic-service systems with 1-limited service hardly any other exact cycle-time results are known. Only for the special case of two completely symmetric queues, an exact formula for the LST of the cycle-time distribution has been obtained [1]. In the present section we extend this result to the asymmetric case. We are thus able to compare $E\mathbf{C}_1^2$ and $E\mathbf{C}_2^2$, and also to determine

$$E\mathbf{C}_{b,i} := E[\mathbf{C}_i \mid A_i], \tag{5.1}$$

with A_i the indicator function of the event 'the cycle contains a service at Q_i'. This quantity plays an important role in several mean waiting-time approximations [2,3,10,14]. Generally speaking, exact cycle-time formulas for the two-queue case give more insight into the accuracy of general approximations for cycle-time distributions, as were proposed by Hashida and Ohara [13] and Kuehn [14].

We now derive an exact expression for the LST of the distribution of $\mathbf{C}_1$;

the analogous result for C_2 is obtained by interchanging all indices. Starting-point of the analysis is the relation

$$E[e^{-\omega C_1}] = F_1(0,0)\, E[e^{-\omega C_1} \mid q_1^{(1)}=0,\ q_2^{(1)}=0] + \tag{5.2}$$
$$[F_1(0,1)-F_1(0,0)]\, E[e^{-\omega C_1} \mid q_1^{(1)}=0,\ q_2^{(1)}>0] +$$
$$[F_1(1,0)-F_1(0,0)]\, E[e^{-\omega C_1} \mid q_1^{(1)}>0,\ q_2^{(1)}=0] +$$
$$[1-F_1(0,1)-F_1(1,0)+F_1(0,0)]\, E[e^{-\omega C_1} \mid q_1^{(1)}>0,\ q_2^{(1)}>0]$$
$$= F_1(0,0)\, \sigma_2(\omega)[\sigma_1(\omega+\lambda_2)+\{\sigma_1(\omega)-\sigma_1(\omega+\lambda_2)\}\beta_2(\omega)] +$$
$$[F_1(0,1)-F_1(0,0)]\, \beta_2(\omega)\sigma_1(\omega)\sigma_2(\omega) +$$
$$[F_1(1,0)-F_1(0,0)]\, \sigma_2(\omega)\times$$
$$[\beta_1(\omega+\lambda_2)\sigma_1(\omega+\lambda_2)+\{\beta_1(\omega)\sigma_1(\omega)-\beta_1(\omega+\lambda_2)\sigma_1(\omega+\lambda_2)\}\beta_2(\omega)] +$$
$$[1-F_1(0,1)-F_1(1,0)+F_1(0,0)]\, \beta_1(\omega)\sigma_1(\omega)\beta_2(\omega)\sigma_2(\omega).$$

$F_1(0,1)$ is given by (3.30). Hence $E[e^{-\omega C_1}]$ can be expressed in $F_1(0,0)$ and $F_1(1,0)$. Substitution of $z_1=1$, $z_2=0$ into (3.5) leads to a linear relation between those two terms:

$$F_1(1,0) = F_1(0,0)\frac{\beta_1(\lambda_2)-1}{\beta_1(\lambda_2)} + \frac{1-\lambda_2 s/(1-\rho)}{\beta_1(\lambda_2)\sigma_1(\lambda_2)}. \tag{5.3}$$

Differentiation of the expressions in (5.2) w.r.t. ω, and substitution of $F_1(1,0)$ into $F_1(0,0)$ using (5.3), leads to cycle-time moments. A simple calculation yields the mean cycle time given in (2.5); a lengthy calculation yields

$$EC_1^2 = s^{(2)} + \sum_{i=1}^{2} \lambda_i \beta_i^{(2)} \frac{s}{1-\rho} + 2\beta_1(\beta_2+s)\frac{\lambda_1 s}{1-\rho} + \tag{5.4}$$
$$2\beta_2 s - 2\beta_2 s_2(1-\frac{\lambda_2 s}{1-\rho}) + 2\beta_2(1-\frac{\lambda_2 s}{1-\rho})[\frac{\beta_1'(\lambda_2)}{\beta_1(\lambda_2)} + \frac{\sigma_1'(\lambda_2)}{\sigma_1(\lambda_2)}] -$$
$$2\beta_2 F_1(0,0)\sigma_1(\lambda_2)\frac{\beta_1'(\lambda_2)}{\beta_1(\lambda_2)}.$$

Note that if the switching time from Q_1 to Q_2 is a constant (s_1), then EC_1^2 only depends on the individual mean switching times via the term involving $F_1(0,0)\sigma_1(\lambda_2)$ - apart from that term, only s and $s^{(2)}$ occur. We'll return to this point in the next section.

We now turn to the cycle-time distribution of C_1 under the condition, A_1, that the cycle contains a service at Q_1. Similarly as (5.2),

$$E[e^{-\omega C_1} \mid A_1] = \frac{F_1(1,0)-F_1(0,0)}{1-F_1(0,1)}\sigma_2(\omega)\times \tag{5.5}$$
$$[\beta_1(\omega+\lambda_2)\sigma_1(\omega+\lambda_2)+\{\beta_1(\omega)\sigma_1(\omega)-\beta_1(\omega+\lambda_2)\sigma_1(\omega+\lambda_2)\}\beta_2(\omega)] +$$

$$\frac{1-F_1(0,1)-F_1(1,0)+F_1(0,0)}{1-F_1(0,1)}\beta_1(\omega)\sigma_1(\omega)\beta_2(\omega)\sigma_2(\omega).$$

A simple calculation leads to

$$\begin{aligned} E\mathbf{C}_{b,1} &= E[\mathbf{C}_1 \mid A_1] \\ &= \beta_1 + s + \beta_2[1-\frac{1-\rho}{\lambda_1 s}\{1-\frac{\lambda_2 s}{1-\rho}-F_1(0,0)\sigma_1(\lambda_2)\}]. \end{aligned} \tag{5.6}$$

A similar expression, with all indices interchanged, holds for $E\mathbf{C}_{b,2}$. Note that the term between curly brackets represents the difference between the probability that S finds Q_2 empty and the probability that S finds first Q_1 and then Q_2 empty. Also observe that $\lambda_1 s/(1-\rho)$ is the probability that S does serve at Q_1. Hence the term between square brackets represents the conditional probability that S does serve at Q_2, under the condition A_1.

6. Numerical analysis

The present section is devoted to a numerical evaluation of some important performance measures of the alternating-service model. Our reasons for including this section are twofold:

(i) We want to show that the BVP formulation leads to formulas which can be numerically evaluated in a straightforward manner;

(ii) We want to contribute to the insight into the behavior of the alternating-service model and, more generally, of cyclic-service models; in particular, the numerical results to be presented may be helpful for devising and testing approximations.

For the sake of (i), we now discuss the numerical evaluation of $F_i(0,0)$ and $E\mathbf{w}_i$; other performance measures are easily evaluated from these quantities. The numerical analysis basically consists of five steps. For details we refer to Ch. IV.1 of [7], in which numerical calculations of this kind have been extensively discussed.

Step 1: Solving Theodorsen's integral equation (cf. (3.15))
Determine $\theta(\phi)$, iteratively, from (cf. [11]):

$$\theta_0(\phi) = \phi, \quad 0 \leqslant \phi \leqslant 2\pi, \tag{6.1}$$

$$\theta_{n+1}(\phi) = \phi - \frac{1}{2\pi}\int_0^{2\pi}\log\{\frac{\delta(\theta_n(\omega))}{\cos(\theta_n(\omega))}\}\,\mathrm{cotan}\{\tfrac{1}{2}(\omega-\phi)\}d\omega, \quad 0 \leqslant \phi \leqslant 2\pi,$$

where $\delta(\theta_n(\omega))$ is determined from (cf. (3.8)):

$$\delta(\theta_n(\omega)) = 2\sqrt{r_1 r_2}\cos(\theta_n(\omega))\sqrt{\beta(\lambda(1-\delta(\theta_n(\omega))))}, \tag{6.2}$$

using the Newton-Raphson root-finding procedure. In our calculations, the iteration has been continued until the differences between successive iterations of $\theta(\cdot)$ (in the supremum norm) were in absolute value less than 10^{-6}. This required between 6 and 14 iterations.

REMARK 6.1. Due to various symmetry properties we can restrict ourselves in the computations, here and in the sequel, to $\phi \in [0,\pi]$. As various integrands that will have to be computed change more rapidly for ϕ close to 0 than for other values of ϕ, a finer subdivision has been chosen for the interval $[0,\pi/5]$ (20 points) than for the interval $[\pi/5,\pi]$ (40 points). All involved integrals have been evaluated using the repeated trapezoidal rule.

Step 2: Determination of the conformal mapping $f_0(e^{i\phi})$, $0 \leqslant \phi \leqslant 2\pi$
Applying the Plemelj-Sokhotski formula (cf. [7], Formula (I.1.6.4)) to (3.14) yields:

$$\begin{aligned} f_0(e^{i\phi}) &= e^{i\phi} \exp[\log\{\frac{\delta(\theta(\phi))}{\cos(\theta(\phi))}\} \\ &\quad + \frac{1}{2\pi i}\int_0^{2\pi} \log\{\frac{\delta(\theta(\omega))}{\cos(\theta(\omega))}\} \operatorname{cotan}\{\tfrac{1}{2}(\omega-\phi)\}d\omega] \\ &= e^{i\theta(\phi)} \frac{\delta(\theta(\phi))}{\cos(\theta(\phi))} \\ &= \delta(\theta(\phi))[1+i \tan(\theta(\phi))], \quad 0 \leqslant \phi \leqslant 2\pi; \end{aligned} \tag{6.3}$$

this result could also have been derived from the formula below (3.8).

Step 3: Determination of $f(2r_2)$ *and* $f^{(1)}(2r_2)$
Using (3.14), $f(2r_2)$ is obtained as the solution, on $[0,1]$, of $f_0(z) = 2r_2$. Again we have used the Newton-Raphson root-finding procedure. $f^{(1)}(2r_2)$ can be obtained in two ways:
(i) by numerical differentiation of $f_0(\cdot)$; note that

$$f^{(1)}(2r_2) = \frac{1}{f_0^{(1)}(2r_2)}; \tag{6.4}$$

(ii) by a numerical evaluation of the expression:

$$f_0^{(1)}(2r_2) = \frac{2r_2}{f(2r_2)} + 2r_2 \frac{1}{2\pi}\int_0^{2\pi} \log\{\frac{\delta(\theta(\omega))}{\cos(\theta(\omega))}\} \frac{2e^{i\omega}}{(e^{i\omega}-f(2r_2))^2}\, d\omega, \tag{6.5}$$

and substitution of the result in (6.4).
For a discussion of (ii) see [7], p. 351. We have used both (i) and (ii), but due to the fact that we have chosen a relatively fine subdivision we have found no significant differences.

Step 4: Calculation of $H(e^{i\phi})$, $0 \leqslant \phi \leqslant 2\pi$
$H(e^{i\phi})$ is obtained from (3.19) by noting that $\operatorname{Re} f_0(e^{i\phi}) = \delta(\theta(\phi))$:

$$\begin{aligned} H(e^{i\phi}) &= -\beta_2(\lambda(1-\delta(\theta(\phi))))\, \sigma_1(\lambda(1-\delta(\theta(\phi))))\frac{2r_1}{f_0(e^{i\phi})} \times \\ &\quad \frac{f_0(e^{i\phi})/2r_1 - \beta_1(\lambda(1-\delta(\theta(\phi))))}{f_0(e^{-i\phi})/2r_2 - \beta_2(\lambda(1-\delta(\theta(\phi))))}, \quad 0 \leqslant \phi \leqslant 2\pi. \end{aligned} \tag{6.6}$$

Step 5: Determination of $E\mathbf{w}_i$ *and* $F_i(0,0)$, $i=1,2$

Once we have calculated $\{\frac{d}{dz}F_1(0,z)\}_{z=1}$ from (4.5), we can obtain $E\mathbf{w}_2$ (cf. (4.3), (4.4) and (4.5)), and subsequently $E\mathbf{w}_1$ from (4.1). $F_1(0,0)$ is easily calculated from (3.32); $F_2(0,0)$ is obtained from (cf. (3.29)):

$$F_2(0,0) = F_1(0,0) \exp[\frac{1}{2\pi} \int_0^{2\pi} \log H(e^{i\omega})\, d\omega]. \tag{6.7}$$

With the subdivision we have chosen, each row in the tables below takes about 15 sec. of CPU time on a Cyber 170 model 750, with very small memory requirements. Using fewer $\theta(.)$ iterations and a less fine subdivision of the interval $[0,\pi]$ leads to a considerable reduction of CPU time, without sacrificing too much accuracy. The computer program was written in Pascal.

Numerical results are presented in Tables I and II. The performance measures under consideration are the mean waiting times $E\mathbf{w}_i$, the second moments of cycle times $E\mathbf{C}_i^2$, the conditional first moments of cycle times $E\mathbf{C}_{b,i}$ (cf. (5.1) and (5.6)) and the empty-system probabilities at server-arrival epochs, $F_i(0,0)$. Table I studies the influence of the switching times on these performance measures. Table Ia shows that the choice of switching-time *distributions* has hardly any effect on $E\mathbf{C}_{b,i}$ and $F_i(0,0)$, and only has a considerable effect on $E\mathbf{w}_i$ and $E\mathbf{C}_i^2$ when mean service times are relatively small. Exactly the same statement can be made concerning the choice of s_1 and s_2, for given *total* mean switching time $s=s_1+s_2$. In Table Ib results for $\beta_1=\beta_2=0.8$ are printed, exhibiting almost-insensitivity for the choice of s_1 and s_2. In the (non-printed) case with $\beta_1=\beta_2=0.2$ and all other parameter values as in Table Ib, the largest difference (with respect to $E\mathbf{w}_i$ and $E\mathbf{C}_i^2$) due to changes in s_i is in the order of 25%. For deterministic switching-time distributions, $E\mathbf{w}_i$, $E\mathbf{C}_i^2$ and $E\mathbf{C}_{b,i}$ appear to be completely independent of s_1 and s_2, given their sum s. The structure of (5.4) and (5.6) shows that the same must hold for $F_1(0,0)\,\sigma_1(\lambda_2) =$ Pr$\{S$ *finds first* Q_1 *and then* Q_2 *empty*$\}$. The robustness of the model for switching times is also being expressed by the pseudo-conservation law for mean waiting times mentioned in Section 4 (cf. (4.1) for the alternating-service model). In the pseudo-conservation law, the expression for a weighted sum of mean waiting times is seen to depend on the switching-time distributions only through the mean s and the second moment $s^{(2)}$ of the *total* switching time - and the influence of the factor involving $s^{(2)}$ is usually small.

Table II presents mean waiting times and cycle-time moments for three different combinations of service-time distributions, viz.:

Case A: both service-time distributions are negative exponential;

Case B: both service-time distributions are hyperexponential distributions with squared coefficient of variation 4 ($H_2(4)$) and balanced means [18];

Case C: $B_1(.)$ is a $H_2(4)$ distribution with balanced means, and $B_2(.)$ is deterministic.

Out of a wide range of distributions and parameter values, we have tried to make a representative choice. The observations from Table I allow us to restrict ourselves to constant switching times, with $s_1 = s_2$. In all cases considered, $s = 0.2$. We discuss all tabulated performance measures in turn.

(i) $E\mathbf{w}_i$
In [3] and [2], a mean waiting-time approximation has been proposed for a cyclic-service model without and with switching times, respectively. It is first argued that

$$E\mathbf{w}_i = \frac{E\mathbf{rc}_i}{1-\lambda_i E\mathbf{C}_{b,i}}, \tag{6.8}$$

with $E\mathbf{rc}_i$ the mean residual cycle time for Q_i. In fact, this is not an exact result. In the alternating-service model it appears to be quite close in most cases, but there are a few exceptions. Taking $E\mathbf{rc}_i = E\mathbf{C}_i^2/2E\mathbf{C}_i$ (acting as if the cycle-time process is a renewal process), formula (6.8) would imply that $E\mathbf{w}_i$ changes linearly with $E\mathbf{C}_i^2$ for fixed first moments of service times and switching times. The table entries for these two quantities suggest that this is indeed more or less the case.

In [2] and [3] two approximation assumptions are introduced to estimate the unknown $E\mathbf{rc}_i$ and $E\mathbf{C}_{b,i}$, viz.:

Assumption 1: $E\mathbf{C}_{b,i} = E\tilde{\mathbf{C}}_{b,i} := (\beta_i + s)/(1-\rho+\rho_i)$
(this approximation is due to KUEHN [14]).

Assumption 2: $E\mathbf{rc}_i$ *is the same for all i.*
Subsequently the pseudo-conservation law (cf. (4.1) for the alternating-service model) is used to estimate the one unknown $E\mathbf{rc}_1$. Below we investigate the accuracy of these assumptions for the alternating-service model.

(ii) $E\mathbf{C}_i^2$
Again taking $E\mathbf{rc}_i = E\mathbf{C}_i^2/2E\mathbf{C}_i$, Assumption 2 above would imply that all $E\mathbf{C}_i^2$ are the same. Indeed, in all considered cases, $E\mathbf{C}_1^2$ and $E\mathbf{C}_2^2$ differ less than 7% (and usually much less). FUHRMANN and WANG [10] suggest another mean waiting-time approximation along similar lines as [2], but they assume that

$$E\mathbf{C}_1^2/E\mathbf{C}_2^2 \approx E\mathbf{C}_{b,2}/E\mathbf{C}_{b,1}; \tag{6.9}$$

our numerical results show that this assumption is not accurate for the alternating-service model. Still, Fuhrmann and Wang improve upon [2] in case of heavy traffic. It is not yet fully clear whether (6.9) becomes more accurate when the number of queues is larger, or whether (6.9) counteracts an inaccuracy in (6.8) or in the approximation for $E\mathbf{C}_{b,i}$.

TABLE I
Mean waiting times and cycle-time moments for the alternating-service model; influence of the switching times

$S_1(.)$	$S_2(.)$	EW_1	EW_2	EC_1^2	EC_2^2	$EC_{b,1}$	$EC_{b,2}$	$F_1(0,0)$	$F_2(0,0)$
det	det	0.236	0.192	0.082	0.082	0.425	0.461	0.796	0.829
det	exp	0.265	0.221	0.093	0.095	0.427	0.462	0.797	0.827
exp	det	0.268	0.216	0.094	0.092	0.426	0.465	0.796	0.830
exp	exp	0.297	0.244	0.104	0.105	0.427	0.466	0.797	0.829
det	det	16.566	2.158	2.100	2.007	1.317	1.745	0.286	0.300
det	exp	16.671	2.172	2.110	2.020	1.318	1.746	0.287	0.299
exp	det	16.674	2.169	2.113	2.017	1.317	1.746	0.286	0.300
exp	exp	16.779	2.183	2.124	2.031	1.318	1.747	0.287	0.300

TABLE Ia. *The influence of the switching-time distributions.* $B_i(.)$ *negative exponential,* $i=1,2$; *in the first four rows* $\beta_1=\beta_2=0.2$, *in the last four* $\beta_1=\beta_2=0.8$. $\lambda=1$, $r_1=0.7$, $s_1=s_2=0.1$.

$S_1(.)$	$S_2(.)$	s_1	s_2	EW_1	EW_2	EC_1^2	EC_2^2	$EC_{b,1}$	$EC_{b,2}$	$F_1(0,0)$	$F_2(0,0)$
det	det	0.05	0.15	16.566	2.158	2.100	2.007	1.317	1.745	0.282	0.310
det	det	0.1	0.1	16.566	2.158	2.100	2.007	1.317	1.745	0.286	0.300
det	det	0.15	0.05	16.566	2.158	2.100	2.007	1.317	1.745	0.290	0.289
det	exp	0.05	0.15	16.802	2.189	2.124	2.037	1.319	1.747	0.283	0.309
det	exp	0.1	0.1	16.671	2.172	2.110	2.020	1.318	1.746	0.287	0.299
det	exp	0.15	0.05	16.592	2.161	2.102	2.010	1.317	1.745	0.290	0.289
exp	det	0.05	0.15	16.593	2.161	2.103	2.010	1.317	1.745	0.282	0.310
exp	det	0.1	0.1	16.674	2.169	2.113	2.017	1.317	1.746	0.286	0.300
exp	det	0.15	0.05	16.808	2.183	2.130	2.031	1.318	1.748	0.290	0.290
exp	exp	0.05	0.15	16.829	2.192	2.127	2.040	1.319	1.747	0.283	0.310
exp	exp	0.1	0.1	16.779	2.183	2.124	2.031	1.318	1.747	0.287	0.300
exp	exp	0.15	0.05	16.834	2.187	2.133	2.034	1.318	1.748	0.291	0.290

TABLE Ib. *The influence of* s_1 *and* s_2 *for given* $s=s_1+s_2$. $B_i(.)$ *negative exponential,* $i=1,2$; $\beta_1=\beta_2=0.8$. $\lambda=1$, $r_1=0.7$.

(iii) $EC_{b,i}$ and $E\tilde{C}_{b,i}$

In all considered cases, the approximation $EC_{b,1} \approx E\tilde{C}_{b,1}$ (see Assumption 1 above) is extremely accurate. The approximation $EC_{b,2} \approx E\tilde{C}_{b,2}$ is much less accurate: the flow-balancing argument on which the approximation is based, should not be applied to the situation of a rarely occurring cycle C_2 with a - sometimes large - service time at Q_2. The approximation becomes useless in the cases marked with an asterisk, because $EC_{b,2}$ exceeds the obvious upper bound $\beta_1+\beta_2+s$; in those cases we have printed the latter number.

Finally we observe that $EC_{b,i}$ is hardly dependent on the choice of the service-time distributions.

(iv) $F_i(0,0)$

$F_i(0,0)$, too, appears to be hardly dependent on the choice of the service-time distributions.

TABLE II

Mean waiting times and cycle-time moments for the alternating-service model

r_1	β_1	β_2	EW_1	EW_2	EC_1^2	EC_2^2	$EC_{b,1}$	$E\tilde{C}_{b,1}$	$EC_{b,2}$	$E\tilde{C}_{b,2}$	$F_1(0,0)$	$F_2(0,0)$
0.7	0.2	0.2	0.236	0.192	0.082	0.082	0.425	0.426	0.461	0.465	0.796	0.829
0.7	0.2	0.5	0.387	0.330	0.139	0.142	0.478	0.471	0.789	0.814	0.772	0.811
0.7	0.2	0.8	0.726	0.599	0.248	0.256	0.554	0.526	1.114	1.163	0.743	0.786
0.7	0.5	0.2	0.738	0.440	0.237	0.231	0.743	0.745	0.629	0.615	0.734	0.759
0.7	0.5	0.5	1.152	0.671	0.369	0.362	0.824	0.824	0.999	1.077	0.690	0.720
0.7	0.5	0.8	2.131	1.098	0.618	0.612	0.932	0.921	1.361	1.500*	0.630	0.663
0.7	0.8	0.2	2.680	0.977	0.707	0.686	1.062	1.064	0.943	0.909	0.606	0.623
0.7	0.8	0.5	5.088	1.406	1.141	1.096	1.175	1.176	1.351	1.500*	0.494	0.513
0.7	0.8	0.8	16.566	2.158	2.100	2.007	1.317	1.316	1.745	1.800*	0.286	0.300
0.9	0.2	0.2	0.256	0.170	0.081	0.081	0.408	0.408	0.477	0.488	0.767	0.831
0.9	0.2	0.5	0.308	0.208	0.099	0.099	0.422	0.421	0.803	0.854	0.758	0.825
0.9	0.2	0.8	0.400	0.269	0.127	0.129	0.438	0.435	1.121	1.200*	0.749	0.816
0.9	0.5	0.2	1.084	0.416	0.292	0.288	0.714	0.714	0.703	0.727	0.653	0.706
0.9	0.5	0.5	1.279	0.473	0.341	0.336	0.737	0.737	1.050	1.200*	0.633	0.687
0.9	0.5	0.8	1.586	0.557	0.413	0.406	0.762	0.761	1.384	1.500*	0.610	0.664
0.9	0.8	0.2	9.564	0.871	1.197	1.182	1.020	1.020	1.122	1.200*	0.304	0.328
0.9	0.8	0.5	16.161	0.959	1.460	1.420	1.053	1.053	1.459	1.500*	0.215	0.233
0.9	0.8	0.8	43.679	1.077	1.841	1.767	1.087	1.087	1.786	1.800*	0.099	0.108

Case A: $B_i(\cdot)$ *negative exponential*, $i=1,2$; $\lambda=1$, $s_1=s_2=0.1$ *(constant switching times).*

TABLE II (CONT'D)

r_1	β_1	β_2	EW_1	EW_2	EC_1^2	EC_2^2	$EC_{h,1}$	$\tilde{EC}_{h,1}$	$EC_{h,2}$	$\tilde{EC}_{h,2}$	$F_1(0,0)$	$F_2(0,0)$
0.7	0.2	0.2	0.330	0.267	0.114	0.113	0.426	0.426	0.463	0.465	0.796	0.830
0.7	0.2	0.5	0.684	0.547	0.230	0.230	0.481	0.471	0.789	0.814	0.773	0.811
0.7	0.2	0.8	1.546	1.122	0.469	0.470	0.559	0.526	1.112	1.163	0.744	0.785
0.7	0.5	0.2	1.389	0.832	0.435	0.425	0.744	0.745	0.642	0.615	0.735	0.762
0.7	0.5	0.5	2.325	1.273	0.689	0.667	0.828	0.824	1.006	1.077	0.692	0.722
0.7	0.5	0.8	4.660	2.118	1.192	1.154	0.937	0.921	1.363	1.500*	0.632	0.663
0.7	0.8	0.2	5.706	2.073	1.450	1.418	1.063	1.064	0.964	0.909	0.608	0.627
0.7	0.8	0.5	11.124	2.855	2.275	2.188	1.177	1.176	1.361	1.500*	0.496	0.516
0.7	0.8	0.8	37.323	4.258	4.130	3.932	1.319	1.316	1.748	1.800*	0.287	0.301
0.9	0.2	0.2	0.355	0.235	0.112	0.111	0.408	0.408	0.478	0.488	0.767	0.831
0.9	0.2	0.5	0.470	0.309	0.148	0.147	0.422	0.421	0.799	0.854	0.758	0.824
0.9	0.2	0.8	0.689	0.441	0.212	0.211	0.438	0.435	1.115	1.200*	0.749	0.815
0.9	0.5	0.2	2.084	0.795	0.558	0.548	0.714	0.714	0.712	0.727	0.653	0.707
0.9	0.5	0.5	2.509	0.902	0.659	0.637	0.737	0.737	1.050	1.200*	0.633	0.687
0.9	0.5	0.8	3.206	1.069	0.813	0.776	0.762	0.761	1.379	1.500*	0.610	0.664
0.9	0.8	0.2	20.659	1.861	2.564	2.521	1.020	1.020	1.127	1.200*	0.304	0.329
0.9	0.8	0.5	35.133	2.021	3.108	2.988	1.053	1.053	1.460	1.500*	0.215	0.233
0.9	0.8	0.8	95.841	2.248	3.908	3.687	1.087	1.087	1.786	1.800*	0.099	0.107

Case B: *$B_i(\cdot)$ hyperexponential (H_2) with squared coefficient of variation 4 and balanced means; $\lambda=1$, $s_1=s_2=0.1$ (constant switching times).*

r_1	β_1	β_2	EW_1	EW_2	EC_1^2	EC_2^2	$EC_{h,1}$	$\tilde{EC}_{h,1}$	$EC_{h,2}$	$\tilde{EC}_{h,2}$	$F_1(0,0)$	$F_2(0,0)$
0.7	0.2	0.2	0.293	0.235	0.102	0.100	0.425	0.426	0.464	0.465	0.796	0.830
0.7	0.2	0.5	0.400	0.333	0.145	0.143	0.475	0.471	0.794	0.814	0.771	0.813
0.7	0.2	0.8	0.605	0.507	0.220	0.219	0.547	0.526	1.121	1.163	0.741	0.789
0.7	0.5	0.2	1.334	0.795	0.418	0.405	0.743	0.745	0.645	0.615	0.734	0.762
0.7	0.5	0.5	1.862	1.021	0.568	0.540	0.823	0.824	1.015	1.077	0.689	0.725
0.7	0.5	0.8	2.909	1.391	0.815	0.772	0.929	0.921	1.376	1.500*	0.628	0.667
0.7	0.8	0.2	5.602	2.030	1.425	1.388	1.063	1.064	0.966	0.909	0.607	0.628
0.7	0.8	0.5	9.975	2.566	2.067	1.970	1.175	1.176	1.366	1.500*	0.494	0.518
0.7	0.8	0.8	29.029	3.410	3.361	3.157	1.317	1.316	1.752	1.800*	0.286	0.303
0.9	0.2	0.2	0.342	0.226	0.108	0.107	0.408	0.408	0.482	0.488	0.767	0.832
0.9	0.2	0.5	0.382	0.252	0.122	0.121	0.422	0.421	0.812	0.854	0.758	0.826
0.9	0.2	0.8	0.444	0.293	0.143	0.141	0.437	0.435	1.135	1.200*	0.748	0.818
0.9	0.5	0.2	2.061	0.785	0.552	0.541	0.714	0.714	0.718	0.727	0.653	0.708
0.9	0.5	0.5	2.346	0.842	0.619	0.596	0.737	0.737	1.069	1.200*	0.633	0.689
0.9	0.5	0.8	2.728	0.915	0.703	0.666	0.762	0.761	1.407	1.500*	0.610	0.667
0.9	0.8	0.2	20.557	1.851	2.552	2.508	1.020	1.020	1.129	1.200*	0.304	0.329
0.9	0.8	0.5	34.081	1.961	3.021	2.901	1.053	1.053	1.465	1.500*	0.215	0.234
0.9	0.8	0.8	88.887	2.091	3.652	3.431	1.087	1.087	1.789	1.800*	0.099	0.108

Case C: *$B_1(\cdot)$ hyperexponential (H_2) with squared coefficient of variation 4 and balanced means; $B_2(\cdot)$ deterministic; $\lambda=1$, $s_1=s_2=0.1$ (constant switching times).*

ACKNOWLEDGEMENT
The authors are indebted to Dr. J.P.C. Blanc for some useful comments.

REFERENCES

1. O.J. BOXMA (1985). Two symmetric queues with alternating service and switching times. E. GELENBE (ed.). *Performance '84,* North-Holland, Amsterdam, 409-431.
2. O.J. BOXMA, B. MEISTER (1986). Waiting-time approximations for cyclic-service systems with switch-over times. *Performance Evaluation Review 14,* 254-262 (Special issue, Proc. Performance '86).
3. O.J. BOXMA, B. MEISTER (1987). Waiting-time approximations in multi-queue systems with cyclic service. *Performance Evaluation 7,* 59-70.
4. O.J. BOXMA, W.P. GROENENDIJK (1987). Pseudo-conservation laws in cyclic-service systems. *J. Appl. Probab. 24,* 949-964.
5. O.J. BOXMA, W.P. GROENENDIJK (1988). Waiting times in discrete-time cyclic-service systems. *IEEE Trans. Commun. COM-36,* 164-170.
6. J.W. COHEN, O.J. BOXMA (1981). The M/G/1 queue with alternating service formulated as a Riemann-Hilbert problem. F.J. KYLSTRA (ed.). *Performance '81,* North-Holland, Amsterdam, 181-199.
7. J.W. COHEN, O.J. BOXMA (1983). *Boundary Value Problems in Queueing System Analysis,* North-Holland, Amsterdam.
8. M. EISENBERG (1979). Two queues with alternating service. *SIAM J. Appl. Math. 36,* 287-303.
9. G. FAYOLLE, R. IASNOGORODSKI (1979). Two coupled processors: the reduction to a Riemann-Hilbert problem. *Z. Wahrsch. Verw. Gebiete 47,* 325-351.
10. S.W. FUHRMANN, Y.T. WANG (1988). Mean waiting time approximations of cyclic service systems with limited service. P.J. COURTOIS, G. LATOUCHE (eds.). *Performance '87,* North-Holland, Amsterdam, 253-265.
11. D. GAIER (1964). *Konstruktive Methoden der Konformen Abbildung,* Springer Verlag, Berlin.
12. F.D. GAKHOV (1966). *Boundary Value Problems,* Pergamon Press, Oxford.
13. O. HASHIDA, K. OHARA (1972). Line accomodation capacity of a communication control unit. *Review of the Electr. Comm. Laboratories, Nippon Telegraph and Telephone Public Corporation, 20,* 231-239.
14. P.J. KUEHN (1979). Multiqueue systems with nonexhaustive cyclic service. *The Bell System Tech. J. 58,* 671-698.
15. W. SZPANKOWSKI, V. REGO (1987). *Ultimate Stability Conditions for Some Multidimensional Distributed Systems,* Report Dept. of Computer Science, Purdue University.
16. H. TAKAGI (1986). *Analysis of Polling Systems,* MIT Press, Cambridge (MA).
17. H. TAKAGI (1987). A survey of queueing analysis of polling models. *Proceedings of the Third International Conference on Data Communication Systems and their Performance,* Rio de Janeiro.
18. H.C. TIJMS (1986). *Stochastic Modelling and Analysis,* Wiley, New York.

19. K.S. WATSON (1985). Performance evaluation of cyclic service strategies - a survey. E. GELENBE (ed.). *Performance '84*, North-Holland, Amsterdam, 521-533.